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Thm: A mapping $f: X \to Y$ is continuous iff $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for any set $B \subset Y$

Proof:- Let $(X, \mathcal{J})$ and $(Y, \mathcal{U})$ be any two topological spaces. Let $f: X \to Y$ be a continuous map.
Let $B \subset Y$ be arbitrary.
To prove that $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$
Since $B \subset \overline{B}$ ∴ we have

$$\boxed{\overline{f^{-1}(B)} \subset \overline{f^{-1}(\overline{B})}} \quad \text{—— (1)}$$

$\overline{B}$ is closed in Y, f is continuous $\Rightarrow f^{-1}(\overline{B})$ is closed in X.

$$\Rightarrow \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B}) \quad \text{—— (2)} \quad \left(\because A = \overline{A} \text{ if } A \text{ is closed} \right)$$

from (1) & (2)

$$\overline{f^{-1}(B)} \subset f^{-1}(\overline{B}) \quad \text{Prove} \quad \text{—— (3)}$$

Now we shall prove that f is continuous map
Let $C \subset Y$ be a closed set, then $C = \overline{C}$ —(4)
By hypothesis

$$\overline{f^{-1}(C)} \subset f^{-1}(\overline{C}) \quad \text{—— (5)}$$

using (4) we get $\overline{f^{-1}(C)} \subset f^{-1}(C)$
But $f^{-1}(C) \subset \overline{f^{-1}(C)}$ for any set C —(6)
Combining (5) & (6) we get

$$f^{-1}(C) = \overline{f^{-1}(C)}$$

which shows that $f^{-1}(C)$ is closed in X
any $C \subset Y$ is closed $\Rightarrow f^{-1}(C)$ is closed in X
f is a continuous map

(16)

Th: 16 A function $f:(X,\mathcal{J}) \to (Y,\mathcal{U})$ is continuous iff

$$[f^{-1}(B)]^{\circ} \supset f^{-1}(B^{\circ}),\quad B \subset Y$$

or $f^{-1}(B^{\circ}) \subset [f^{-1}(B)]^{\circ}$

Proof: Let $f:(X,\mathcal{J}) \to (Y,\mathcal{U})$ be a topological space mapping. Let $B \subset Y$ be arbitrary.

(i) Suppose f is continuous.
To prove that $[f^{-1}(B)]^{\circ} \supset f^{-1}(B^{\circ})$

$B \subset Y \Rightarrow B^{\circ}$ is open in Y
$\Rightarrow f^{-1}(B^{\circ})$ is open in X (f is continuous)
$\Rightarrow [f^{-1}(B^{\circ})]^{\circ} = f^{-1}(B^{\circ})$ —— (1)

$B^{\circ} \subset B \Rightarrow f^{-1}(B^{\circ}) \subset f^{-1}(B)$
$\Rightarrow f^{-1}(B) \supset f^{-1}(B^{\circ})$
$\Rightarrow [f^{-1}(B)]^{\circ} \supset [f^{-1}(B^{\circ})]^{\circ} = f^{-1}(B^{\circ})$

$\Rightarrow [f^{-1}(B)]^{\circ} \supset f^{-1}(B^{\circ})$ —— (2)

(ii) Suppose $[f^{-1}(B)]^{\circ} \supset f^{-1}(B^{\circ})$
To prove that f is continuous.
Let G be an open subset of Y then $G = G^{\circ}$
If we show that $f^{-1}(G)$ is open in X.

$[f^{-1}(G)]^{\circ} \supset f^{-1}(G^{\circ})$ (by (2))
$\Rightarrow f^{-1}(G)$

$\therefore [f^{-1}(G)]^{\circ} \supset f^{-1}(G)$

But $[f^{-1}(G)]^{\circ} \subset f^{-1}(G)$ is always true
Combining last two $[f^{-1}(G)]^{\circ} = f^{-1}(G)$
$\Rightarrow f^{-1}(G)$ is open in X